

The continuum problem in microscopic phenomena

O. Yaremchuk

June 19, 2014

Abstract

The undecidability of the continuum hypothesis should be understood as undecidability of the set of intermediate cardinality in continuum that means uncertainty of the set members' locations.

The continuum problem has an intuitive statement and solution that leads to the same conclusion.

Quantum phenomena can be explained by decrease of the number of points in microscopic regions.

The set of intermediate cardinality can be produced in a natural way by removal of excess points from a continuous interval which reduces to the fission of the continuous interval into a few subintervals of intermediate cardinality. The simplest form of the fission gives objects analogous to the basic objects of the standard model.

It is assumed that increase of cardinality of cosmological intervals might give phenomena analogous to that of dark matter and dark energy.

Frequency definition of probability, as an experimental law, is considered as evidence of actual infinity.

1 Introduction

Two sets X and Y have the same cardinal number ($|X| = |Y|$) if there exists a one-to-one correspondence $X \leftrightarrow Y$ between their members. The inequality of the cardinal numbers is defined as follows: $|X| < |Y|$ if $X \subset Y$ and $|X| \neq |Y|$. The concept of cardinal number is valid for the infinite sets as well as for the finite ones. In 1870's G. Cantor discovered that there are only two infinite cardinal numbers: any infinite set is equivalent either to the set of the natural numbers N or to the set of the real numbers R . He failed to find

a set M that had more members than N but less than R ($|N| < |M| < |R|$) and supposed that such a set did not exist. This statement is known as the continuum hypothesis (CH) [1].

The continuum problem was raised in the framework of Cantor's naïve set theory. Formalization of set theory, the standard Zermelo-Fraenkel axiomatics, allowed K. Gödel and P. Cohen to establish the independence (undecidability) of the continuum hypothesis [2, 3], which was understood as impossibility to prove or disprove *existence* of the set of intermediate cardinality. Therefore, status of the intermediate set is still regarded as unclear.

2 The status of the set of intermediate cardinality

However, the independence of CH determines the definite status of the set of intermediate cardinality without any additional axiom. It is important here that the intermediate set is a subset of continuum by definition. Recall also that the intermediate set cannot be constructed: Gödel proved that the set does not exist in his “constructible universe” [2].

In order to avoid extra statements, the undecidability of CH should be understood as undecidability of the intermediate subset M with respect to continuum: for any $a \in R$ the statement $a \in M \subset R$ is undecidable. In other words, the locations of the intermediate subset members in R are uncertain. There is no selection rule by which the intermediate set members can be separated from the other points of continuum.

It should be stressed that this is the only definite status of the intermediate set that follows directly from the independence of the continuum hypothesis without any additional assumption.

3 Physical aspect

Any interval of the real line has the same infinite number of points as the entire line. Fig.1a illustrates this presumably well known fact.

When we move two points of the real line closer together, cardinality of the interval between them remains constant and equal to the left and right cardinalities of the environment (Fig.1b):

$$|\{-\infty A\}| = |\{AB\}| = |\{B\infty\}|. \quad (1)$$

If we suppose that number of points of a space interval can decrease

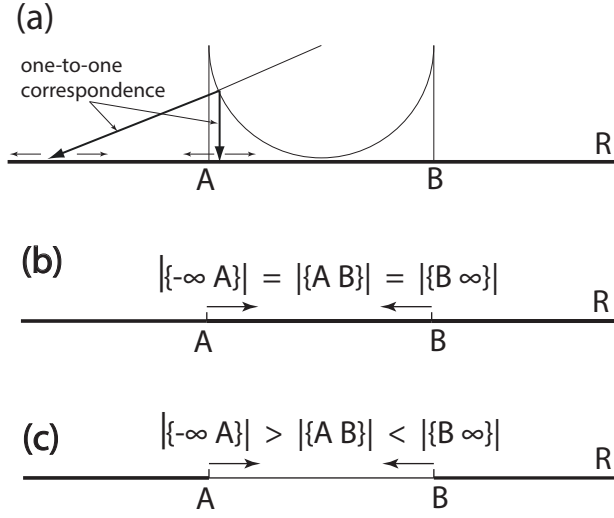


Figure 1: a) Any interval has the same number of points as the entire real line. The sliding line establishes one-to-one correspondence between arbitrarily small interval AB and the real line R . b) Cardinality of the interval AB is equal to the left and right cardinalities of the environment. c) If cardinality of AB decreases, we get the interval of intermediate cardinality.

(Fig.1c), we get a version of the continuum problem, since in this case

$$|R| = |\{-\infty A\}| > |\{AB\}| < |\{B\infty\}| = |R|, \quad (2)$$

i.e., $|N| < |\{AB\}| < |R|$. Recall that the countable set has measure zero. Any extended interval should have more than $|N|$ members.

The interval of intermediate cardinality cannot have definite end points in R , since between any pair of different real numbers there is a complete continuous interval while between any two points of a stand-alone intermediate set should be less than continuum of points. The relative positions of the intermediate set members are different from the relative positions of the real numbers. The intermediate set members are closer to each other in sense of cardinality of the intervals between them.

This means that the intermediate interval cannot have certain length. Absence of the regular length and of the end points makes it unobservable (“dissolved” throughout entire continuum). In particular, the intermediate interval with uncertain length cannot be drawn. Fig.1c visualizes the uncertain interval by giving random length to it which is actually a random mapping of the interval into its continuous environment.

4 Maps

Consider maps of the intermediate set I to the sets of real numbers R and natural numbers N :

$$N \leftarrow I \rightarrow R. \quad (3)$$

The map $I \rightarrow N$ decomposes I into the countable set of mutually disjoint infinite subsets: $I \rightarrow \{I_n\}$ ($n \in N$). All members of I_n have the same countable coordinate n .

Consider the map $I \rightarrow R$. According to the independence of the continuum hypothesis, the map cannot locate the intermediate subset members inside continuum, i.e., there cannot be a universal rule for assigning a unique real number for an arbitrary point of the set of intermediate cardinality (such a rule would be a proof of the negation of CH). In the absence of a general rule for the points localization, the bijection can assign only random (arbitrarily chosen) real number r_{random} to an arbitrary $s(n) \in I$. Thus we get probability $P(r, n)dr$ of finding the point $s(n) \in I$ about r . All real numbers are equiprobable, since there is no reason to prefer any one.

Only the natural number coordinate gives information about relative positions of the points of the intermediate set and, consequently, about sizes of intervals in I but the points with the same countable coordinate are indistinguishable.

For two real numbers a and b the probability $P_{a \cup b}dr$ of finding $s(n)$ in the union of the neighborhoods $(dr)_a \cup (dr)_b$

$$P_{a \cup b}dr \neq [P(a) + P(b)]dr \quad (4)$$

because the point cannot be located in only one of the neighborhoods (the locations are not mutually exclusive while probability is additive for mutually exclusive events only). It is most natural, in this case, to compute the non-additive probability from some additive object by a simple rule. Since the point corresponds to all real numbers simultaneously, we may associate with the point a function $\psi(r)$ defined on the same domain R such that $P(r) = \mathcal{P}[\psi(r)]$ and $\psi_{a \cup b} = \psi(a) + \psi(b)$. It is quite clear that the dependence $\mathcal{P}[\psi(r)]$ should be non-linear. Indeed,

$$P_{a \cup b} = \mathcal{P}(\psi_{a \cup b}) = \mathcal{P}[\psi(a) + \psi(b)] \neq \mathcal{P}[\psi(a)] + \mathcal{P}[\psi(b)]. \quad (5)$$

In order to ensure invariance under shift in I , which is the only requirement besides nonlinearity, we choose the next-to-linear square dependence:

$$\mathcal{P}[\psi(r)] = |\psi(r)|^2. \quad (6)$$

In this case the function ψ is defined up to the factor $e^{i\text{const}}$:

$$\psi(n + \text{const}, r) = e^{i\text{const}} \psi(n, r). \quad (7)$$

Then if ψ is of the form

$$\psi(r, n) = A(r) e^{2\pi i n}, \quad (8)$$

both conditions, nonlinearity and invariance, are satisfied.

Thus the point of the intermediate set may be associated with the function Eq.(8). We can specify the point by the function $\psi(n, r)$ before the mapping and by the random real number and the natural number when the mapping has performed.

5 Probability of transition

Further, it can be shown that the relation between countable and continuous coordinates of an intermediate set point conforms to the basic postulate of the Feinman's formulation of quantum mechanics [4, 2-2].

Consider probability $P(a, b)$ for the point s to go from a at the time t_a to b at t_b :

$$r(t_a), n(t_a) \rightarrow \psi(t) \rightarrow r(t_b), n(t_b), \quad (9)$$

where $t_a < t < t_b$ and $\psi(t) = \psi[n(t), r(t)]$.

Partition interval (t_a, t_b) into k equal parts ε :

$$\begin{aligned} k\varepsilon &= t_b - t_a, \\ \varepsilon &= t_i - t_{i-1}, \\ t_a &= t_0, t_b = t_k, \\ a = r(t_a) &= r_0, b = r(t_k) = r_k. \end{aligned} \quad (10)$$

The conditional probability for the point s to go from $r(t_{i-1})$ to $r(t_i)$ is given by

$$P(r_{i-1}, r_i) = \frac{P(r_i)}{P(r_{i-1})}, \quad (11)$$

i.e.,

$$P(r_{i-1}, r_i) = \left| \frac{A_i}{A_{i-1}} e^{2\pi i \Delta n_i} \right|^2, \quad (12)$$

where $\Delta n_i = |n(t_i) - n(t_{i-1})|$.

The probability of the sequence of the transitions

$$r_0, \dots, r_i, \dots, r_k \quad (13)$$

is given by

$$P(r_0, \dots, r_i, \dots, r_k) = \prod_{i=1}^k P(r_{i-1}, r_i) = \left| \frac{A_k}{A_0} \exp 2\pi i \sum_{i=1}^k \Delta n_i \right|^2. \quad (14)$$

Then we get probability of the corresponding continuous sequence of the transitions $r(t)$:

$$P[r(t)] = \lim_{\varepsilon \rightarrow 0} P(r_0, \dots, r_i, \dots, r_k) = \left| \frac{A_k}{A_0} e^{2\pi i m} \right|^2, \quad (15)$$

where

$$m = \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^k \Delta n_i. \quad (16)$$

Since at any time $t_a < t < t_b$ the point s corresponds to all points of R , it also corresponds to all continuous random sequences $r(t)$ simultaneously, i.e., probability $P[r(t)]$ is non-additive too. Therefore, we introduce an additive functional $\phi[r(t)]$ and, taking into account Eq.(15), get

$$P[r(t)] = |\phi[r(t)]|^2 = |\text{const } e^{2\pi i m}|^2. \quad (17)$$

Thus we have

$$P(a, b) = \left| \sum_{\text{all } r(t)} \text{const } e^{2\pi i m} \right|^2, \quad (18)$$

i.e., the probability $P(a, b)$ satisfies the conditions of Feynman's approach (section 2-2 of [4]) for $S/\hbar = 2\pi m$. Therefore,

$$P(a, b) = |K(b, a)|^2, \quad (19)$$

where $K(b, a)$ is the path integral (2-25) of [4]:

$$K(b, a) = \int_a^b e^{2\pi i m} D r(t). \quad (20)$$

It remains to identify m and action (S/\hbar).

In section 2-3 of [4] Feynman explains how the principle of least action follows from the dependence

$$P(a, b) = \left| \sum_{\text{all } r(t)} \text{const } e^{(i/\hbar)S[r(t)]} \right|^2 \quad (21)$$

if S is very large in relation to \hbar . Since Feynman does not use here that S/\hbar is just action, we can apply the same reasoning to Eq.(18) and, for very large m , get “the principle of least m ”:

$$m = \int_a^b dm(t) = \int_a^b \dot{m}(t) dt = \min. \quad (22)$$

This also means that for large m the point s has a definite path and, consequently, a definite continuous coordinate. In other words, the interval of the intermediate set with the large countable length m is sufficiently close to continuum, i.e., cardinality of the intermediate set depends on its size.

Note that $m(t)$ is a step function and its time derivative $\frac{dm}{dt} = \dot{m}(t)$ is almost everywhere exact zero. But for sufficiently large increment $dm(t)$ the time derivative $\dot{m}(t)$ makes sense as a non-zero value.

The function $m(t)$ may be regarded as a function of $r(t)$: $m(t) = \eta[r(t)]$. It is important that $r(t)$ is not random in the case of large m . Therefore,

$$\int_a^b dm(t) = \int_a^b \frac{d\eta}{dr} \dot{r} dt = \min, \quad (23)$$

where $\frac{d\eta}{dr} \dot{r}$ is a function of r , \dot{r} , and t . This is a formulation of the principle of least action (note absence of higher time derivatives than \dot{r}), i.e., large m can be identified with action.

Since the value of action depends on units of measurement, we need a parameter h depending on units only such that

$$hm = \int_a^b L(r, \dot{r}, t) dt = S, \quad (24)$$

where $L(r, \dot{r}, t) = \frac{d\eta}{dr} \dot{r}$ is the Lagrange function of the point.

Finally, we may substitute S/\hbar for $2\pi m$ in Eq.(20) and get the basic phenomenological postulate of the Feynman’s formulation of quantum mechanics.¹

Note that, if time rate of change of the countable coordinate is not sufficiently high, action vanishes: $\dot{m}(t)$ and, consequently, $dm = \dot{m}(t)dt$ is exact zero. This may be understood as vanishing of the mass of the point. Formally, mass is a consequence of the principle of least action: it appears in the Lagrangian of a free particle as its specific property. Thus mass is analogous to air drag which is substantial only for sufficiently fast bodies.²

¹Note also that above we actually “discovered” classical mechanics.

²It cannot be stated that this way of getting mass contradicts the Higgs and other mechanisms.

We should also expect that that there exist a transitional countable speed for which mass is uncertain.

The countable axis is not an auxiliary concept but an important phenomenon, since it gives action. Feynman could extend his formulation of quantum mechanics, without knowing about the intermediate set, by introducing the discrete motion alongside with the continuous motion and getting action as above, as a result of dequantization of the discrete path. Otherwise, in the original Feynmans approach without the countable coordinate, S/\hbar remains in essence non-quantized, continuous value.

6 The spectrum of the infinite sets

Thus in order to obtain an intermediate set, it is necessary to reduce the number of points of a continuous interval. For this it is sufficient to remove excess points outside the interval. Since elimination of a countable set cannot change cardinality of continuum [5, Ch. 5, §6, Theorem 4], the set of excess points should have intermediate cardinality as well. In other words, we can get the intermediate set by fission of the continuous interval into subintervals of intermediate cardinality.

In the framework of Zermelo-Fraenkel set theory, we can get the intermediate set through the stage of continuum, by the backward step:

$$\begin{array}{ccc} & \xleftarrow{\text{fission}} & \\ N & \overset{\times}{\rightarrow} I & R. \\ & \xleftarrow{2^N} & \end{array} \quad (25)$$

In set theory, infinite sets originate from the finite ones due to the axiom of infinity and the power set axiom (the 2^N -step in the above expression) as a result of the stepwise process of construction starting with the empty set.

However, in reality infinity is rather a hereditary property, i.e., it is impossible to obtain an infinite set by means of any procedure of collecting and processing individual elements and finite sets. An infinite set can be obtained only from another infinite set.

Consider the natural scheme of fission shown in the Fig. 2a.

Decrease of cardinality of the interval should affect its length. Regular length means that the interval is congruent to some complete continuous interval, in the simplest case of the interval of length 1, to the unit interval. Congruence is a geometrical identity, point-by-point coincidence, implying set-theoretic equivalence.

Minimum decrease of cardinality should damage the length: the interval remains extended but its length is uncertain. Sufficiently large decrease

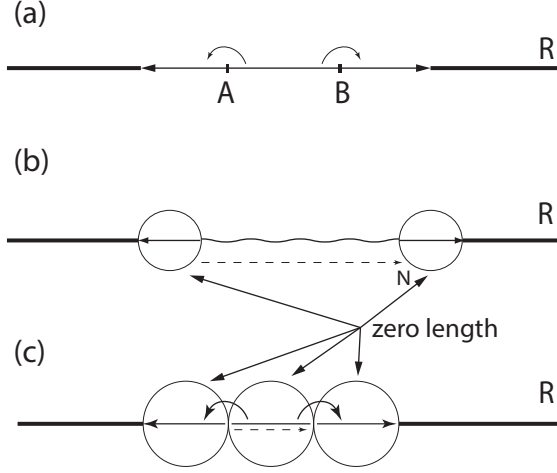


Figure 2: a) The fission of the continuous interval into three intervals of intermediate cardinality. b) The fission into one uncertain and two point-like intervals. c) The fission into three point-like intervals.

should destroy the length: the interval becomes point-like. The location of this point is also uncertain.

If we remove from the continuous interval the minimum sets of intermediate cardinality, the removed (extreme) intervals should have zero-length while the interval between them should be uncertain (Fig. 2 b). Since the point-like intervals are the endpoints of the extended interval, they move in the opposite directions along the countable axis of the middle interval. The interval moving in the negative direction may be called antipoint. In this sense, we have a point-antipoint pair.

We get the following spectrum of infinite sets with respect to length as a grading property:

$$|R| > |I_{\sim}| > |I_0| > |N|, \quad (26)$$

where I_{\sim} and I_0 are the intervals with uncertain and zero lengths respectively. Since non-equivalent intervals cannot be linearly dependent, the one-dimensional axis splits into subaxes with different internal equivalences (symmetries).

From the scheme of the fission, it follows that force is analogous to pressure difference that appears at the border between different cardinalities. As was mentioned above, mass is analogous to air drag. In the case of gas, both drag and pressure are proportional to the surface area which makes them “equivalent” (proportional). Therefore, the equivalence principle may be regarded as an argument for careful use of the analogy for rough comparison of masses and forces.

More correctly, mass is determined by the cumulative countable path of all the points, which may be taken as the area swept by the interval, while force is due to the difference in properties of extension (lengths) on the sides of the point-like interval.

We can try to decompose the continuous interval into three point-like intervals. For this, we should take out more points than a member of I_0 has to make the remaining interval point-like. The extracted intervals should still have zero length as well (Fig.2 c).

One more set of proper microscopic intervals gets into the spectrum of infinite cardinalities:

$$|R| > |I_{\sim}| > |I_0^2| > |I_0^1| > |N|. \quad (27)$$

If all three intervals are point-like, we get a more complex point consisting of three inseparable composite points which we preliminarily consider as members of I_0^2 . In this case, for the countably motionless middle interval we also have the point-antipoint pair but separated by the zero-length interval. Fig. 3 shows that the configurations of the complex point-like object has the obvious analogy with the meson and nucleon structures of the Standard Model.

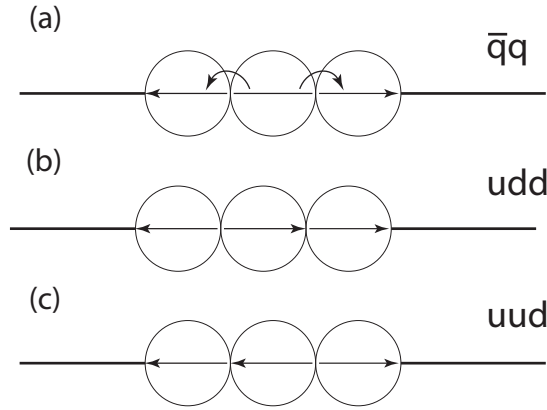


Figure 3: The analogy with the Standard Model. A simplified scheme. Preliminarily, the right-moving intervals are identified with down quarks, since, for the whole object moving to the right (only this case is shown), they are countably faster and should have greater mass.

The following fact should be noted: as a result of the fission we have a complete set of fragments of the empty, structureless continuous interval. This fact may be reformulated into the idea of chromodynamics: colored parts make up colorless whole. Figuratively, this may be called “Pinocchio

principle”: we can number all the parts of Pinocchio (or paint all them in different colors) but, if we could put them together, we would have the initial peace of wood which is not a part of Pinocchio and therefore has no number (or color). Analogously, we may formally assign some properties to each interval of the set of the fission fragments, property 1, property 2, property 3, combination of which gives absence of such a property since the initial empty interval, from which they are “cut out”, is not a member of the set of its fragments.

This principle also works for the poin-antipoint pair production.

In order to destroy the complex point, it is necessary to insert a non-zero length interval between the point-like intervals. This seems to be possible if we take into account that the set of the uncertain intervals splits into two subsets: the set of constantly extended uncertain intervals (always nonzero length) I_{\sim} with U(1) symmetry, which is actually accounted for in Eq.(7), and the set of still unaccounted uncertain intervals whose possible value of length reaches zero (two-fold type of zero-nonzero length) $I_{\sim,0}$:

$$\begin{array}{ccccccc}
 |R| & > & |I'_{\sim}| & > & |I_0^2| & > & |I_0^1| & > & |N| & , & (28) \\
 & & \swarrow & & \searrow & & & & & & \\
 |R| & > & |I_{\sim}| & > & |I_{\sim,0}| & > & |I_0^2| & > & |I_0^1| & > & |N|
 \end{array}$$

where I'_{\sim} is complete set of uncertain intervals $I'_{\sim} = I_{\sim} \cup I_{\sim,0}$.

An interval of the second type, $I_{\sim,0}$, which can spontaneously turn from zero to non-zero length, could just cause decay when it is placed between the other point-like intervals. However, the change from exact zero to non-zero length is an instant displacement implying infinite relative speed of the moving apart components of the complex point and infinite energy, which is impossible whatever small the non-zero interval is. As a result, the interval, in order to fit into the zero-length gap, should rather decay itself by the same scheme (Fig. 4), lose some number of points, and become identically zero length interval.³ The two minimum subintervals are ejected at the non-zero-length speed.

For the moving interval (Fig. 4 a), the decay is asymmetrical, since the whole object is moving to the right at the speed of the basic middle interval. Therefore, using analogy with air drag, the virtual non-zero-length middle interval W gets large mass as a large “windage” in consequence of large “sail

³We should distinguish exact zero length from infinitesimal length. Infinitesimal is a sequence approaching zero. Between any member of the sequence and the limit there is an arbitrarily small but complete continuous interval, i.e., the members of the sequence does not get closer to the limit in sense of cardinality.

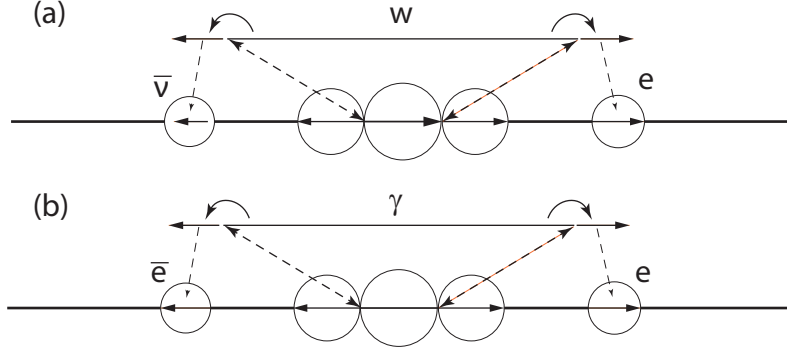


Figure 4: Possible schemes of the decay of the complex point-like object. Symbols point to the analogy with the standard model.

area”. Recall that mass of an intermediate weak boson is much greater than mass of quark.

As a result of the combination of speeds, the countable speed of the left-moving minimum subinterval $\bar{\nu}$ is smaller than that of the right-moving one e . Accordingly, the mass of this antipoint should be smaller than the mass of the right-moving point. The speed might be insufficient to get regular mass, i.e., the antipoint’s mass might be even uncertain (not a number but an observable).

The motionless initial interval gives a symmetrical point-antipoint pair (Fig. 4 b). In the absolutely symmetric case of the two identical intervals with opposite countable motions, close point-antipoint pair forms a countably static, i.e., massless, object which gives “annihilation” or decay through a massless intermediate interval γ (“electromagnetic decay”). If there is some asymmetry in the countable motions, the intermediate “boson” gets mass as a consequence of the resultant non-zero countable speed (“weak decay”).

The second line of Eq. 28 gives more detailed spectrum:

$$|R| > |I_{\sim}| > |I_{\sim,0}| > |I_0^2| > |I_0^1| > |N|. \quad (29)$$

Since the regular intervals of one cardinality are not regular intervals for the other cardinalities, the complete description splits into separate conflicting descriptions. The conflicts have the character of dualities.

7 One more aspect of the physical continuum problem

The physical statement of the continuum problem has, however, one more aspect that should be mentioned. If we ask “can cardinality of a small interval decrease?” it is also natural to ask “can cardinality of a large interval increase?” Can large cosmological intervals have more points than the intervals of our scale and how should it be manifested?

There are facts that might be consequences of existence of higher cardinalities:

1. Two heavier generations of elementary particles might be a result of the fission of the intervals of two greater cardinalities.
2. Since force is analogous to pressure that appear at the border between different cardinalities and gravity acts only as attraction, there must be an external greater cardinality providing the pressure. This might explain why gravity remains outside the standard model: all the interactions of the standard model are due to the internal cardinalities.
3. Microscopic intervals have uncertain or zero lengths since they do not have enough points for all real numbers. Their lengths are, figuratively speaking, “under-inflated” and therefore “formless”. If there are more points than the real numbers, length is rather like an overinflated balloon. More powerful interval cannot be kept within an interval with regular length and should tend outside. It should seem increasing or even increasing with acceleration. Observationally, this might appear as an additional speed or an acceleration of sufficiently distant cosmological objects.

8 Probability as evidence of actual infinity

The continuum problem is physically meaningful only if the space is really an infinite set of points. On the one hand, it is a well-established experimental fact due to successful use of calculus in physics. On the other hand, there is a direct evidence of the existence of infinity. This is a concept or rather phenomenon of probability.

The defining property of an infinite set is equivalence to its proper subset. A proper equivalence of an object is called its symmetry. Cardinal number of the set is determined by the same equivalence. Thus any infinite set has an inherent symmetry determined by its cardinality.

Phenomenon of probability consist in increase of symmetry of the set of the outcomes with increase of the number of random events, when the number approaches infinity. This fact means, in essence, direct impact of

actual infinity: the set of the outcomes gets closer to infinity, experiences its greater influence, and becomes more symmetrical. Space, as the only actual infinite point set, makes its symmetrizing influence. The frequency definition of probability is an experimental law where the approach to infinity is a real process. If infinity produces an impact then it exists.

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